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1983 J. Phys. A: Math. Gen. 16 99

(http://iopscience.iop.org/0305-4470/16/1/017)

J. Phys. A: Math. Gen. 16 (1983) 99–106. Printed in Great Britain

Null geodesics in the static Ernst space–time

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Received 24 May 1982

Abstract. The null geodesics in the equatorial plane of the Ernst space-time have been studied. It is found that they fall essentially into two types depending on the value of the dimensionless magnetic field parameter β. If β is less than $\beta_c = 0.094\,68$ then there exist persisting orbits and stable and unstable circular orbits. However, for $\beta > \beta_c$ there are no persisting orbits and every particle with non-zero angular momentum must fall into the singularity. For $\beta > 0$ no particle with non-zero angular momentum can escape to infinity. The weak field case $\beta \ll 1$ has been treated in detail and the radii of the stable and unstable circular orbits have been obtained in this limit. Finally, the gravitational redshift has been calculated relative to static observers.

1. Introduction

The study of geodesic motion in the Schwarzschild, Kerr, Kerr–Newman, Nordstorm and Ernst fields representing a static Schwarzschild-type black hole immersed in an external, axially symmetric, magnetic field has created considerable interest in this subject (see Darwin 1959, 1961, Carter 1966, 1968, De Felice 1968, De Felice and Calvani 1972, Dadhich and Kale 1977, Dadhich *et al* 1979, Dadhich and Wiita 1981). Dadhich *et al* (1979) have studied the motion of charged particles in Ernst's static space–time (1976) in which the magnetic field becomes uniform asymptotically if $|Bm| \ll 1$. In the special case of absence of mass in this space–time the solution reduces to Melvin's magnetic universe (1964). Melvin and Wallingford (1965) have examined geodesics in this universe.

The present work aims at investigating the behaviour of null geodesics in the Ernst space–time. The study of null geodesics is of great interest in probing into the geometry of the space–time and the gravitational field associated with it. The stability of a Schwarzchild black hole immersed in a magnetic field is of considerable interest, and hence our examination of the geometry may to some extent be a step in that direction. One could categorically make a statement whether the geodesics in the Ernst space–time are more bound than that of the Schwarzschild case. If the geodesics are pulled closer, due to the presence of the magnetic field, then this could possibly suggest that we are dealing with the gravitationally more stable system.

In the classical limit, zero mass particles, like photons, neutrinos, etc travel along null geodesics and hence our investigations have some astrophysical relevance. One of the motivating factors is the study of the optical appearance of compact objects

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immersed in strong magnetic fields. Such situations might exist near the centres of galaxies. The current black hole dynamo models consider external fields to play an important role in energy production mechanisms. Our work will also go towards determining the energy flux profiles of neutrinos emitted from the vicinity of the compact objects. In general, the problem of radiation propagation can be examined in this light as the radiation tends to travel along null geodesics. The magnetic field will alter the course of the zero mass particles from that of the Schwarzschild case.

Our investigations show that the gravitational field of the Ernst space–time is extremely strong. This gravitational field can be compared with the two well known limiting cases, namely, the Schwarzschild case ($B = 0$) and the Melvin magnetic universe ($m = 0$) case with the aid of null geodesics. The null geodesic in the Schwarzschild case can almost always escape to infinity, implying that the gravitational attraction is limited. In the case of Melvin's magnetic universe, the parallel bundle of magnetic field lines produces a gravitational field so strong that every particle with non-zero angular momentum moves in a bound orbit. In the Ernst case the gravitational field is the sum total of the two fields due to both the central mass and the magnetic field and therefore is immensely powerful. The gravitational field of the Ernst space–time retains the features of the horizon at $r = 2m$ and the singularity at $r = 0$ of the Schwarzschild space–time, and at the same time possesses the characteristics of the Melvin magnetic universe for large radial distances. The gravitational field falls off more slowly than that of the Schwarzschild black hole. Due to this reach of the magnetic field, all particles with non-zero impact parameter cannot escape to infinity but can only move in bound orbits. In fact, for sufficiently strong magnetic fields $\beta \geqslant 0.09468$, where $\beta = 8.5 \times 10^{-7}\,(M/M_{\odot})(B/10^{12}\,\text{gauss})$ is a dimensionless parameter which gauges the strength of the magnetic field, $M_{\odot}$ is the mass of the sun and B the magnetic flux intensity, every particle with non-zero impact parameter must fall into the singularity. For β less than this critical value there also exist persisting orbits. As in the case of Melvin's magnetic universe and unlike the Schwarzschild case, a particle with non-zero impact parameter is never able to attain an asymptotically straight trajectory. Finally, the redshift for the particles has been computed relative to static observers. The redshift plays a crucial role in energy considerations.

In § 2 we obtain the first integrals by the Hamiltonian–Jacobi methods. In § 3 we discuss the effective potential obtained and the null trajectories. In § 4 we take up the weak magnetic field case of $\beta \ll 1$. In § 5 we compute the gravitational redshift associated with the null trajectories.

2. The equations of motion

The Ernst space–time which represents a Schwarzschild black hole immersed in an axially symmetric magnetic field is described by the metric in the units of $c = G = 1$ by

$$\mathrm{d}s^2 = \Lambda^2\left[\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 - \left(1-\frac{2m}{r}\right)\mathrm{d}t^2\right] + \frac{r^2\sin^2\theta}{\Lambda^2}\,\mathrm{d}\varphi^2 \tag{2.1}$$

where $\Lambda = 1 + B^2 r^2 \sin^2\theta$. The quantities m and B represent the mass and the constant value of the magnetic field on the axis respectively. The magnetic field becomes uniform asymptotically. The metric (2.1) reduces to the Schwarzchild solution for $B = 0$ and to the Melvin magnetic universe when $m = 0$.

Following Carter (1968), we can write down the Hamilton–Jacobi equation for the function S:

$$\partial S/\partial \lambda = \tfrac{1}{2} g^{ij} (\partial S/\partial x^i) \partial S/\partial x^j \tag{2.2}$$

where λ is an affine parameter along the geodesic. Substituting the value of g^{ij} obtained from (2.1), the equation (2.2) for the massless particle assumes the form

$$0 = \Lambda^{-2}\left(1-\frac{2m}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{\Lambda^2 r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\Lambda^2}{r^2 \sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 - \Lambda^{-2}\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{\partial S}{\partial t}\right)^2. \tag{2.3}$$

Since the coordinates t and φ do not appear explicitly in the metric we can immediately write down a form for S:

$$S = S^*(r, \theta) + l\varphi - Et \tag{2.4}$$

where l and E are constants representing respectively the angular momentum and the energy of the particle. Unfortunately the equation (2.3) is not separable in general in r and θ coordinates. However, if we set $\theta = \pi/2$ and $\mathrm{d}\theta/\mathrm{d}\lambda = 0$, the equation becomes separable and the first integrals may thus be obtained. It is also evident from symmetry considerations that the particle will continue to travel in the plane $\theta = \pi/2$. The first integrals in the case of the equatorial plane are given by

$$\dot\varphi = l\Lambda^2/r^2 \sin^2\theta \tag{2.5}$$

$$\dot t = E/\Lambda^2(1-2m/r). \tag{2.6}$$

The dot over a quantity denotes differentiation with respect to the affine parameter. Since the geodesic is null, another first integral is given by $ds = 0$. In the light of these equations, the radial propagation is governed by the equation

$$\dot r^2 = \frac{(1-2m/r)}{\Lambda^2}\left(\frac{E^2}{\Lambda^2(1-2m/r)} - \frac{l^2\Lambda^2}{r^2 \sin^2\theta}\right). \tag{2.7}$$

The equations (2.5), (2.6) and (2.7) describe the null geodesics in the equatorial plane completely. From these equations we may determine the quantity $\mathrm{d}r/\mathrm{d}\varphi$, which gives the radial displacement relative to the angular displacement,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 = \frac{r^4}{\Lambda^8}\left[\frac{1}{b^2} - \frac{\Lambda^4}{r^2}\left(1-\frac{2m}{r}\right)\right] \tag{2.8}$$

where $b = l/E$ is the impact parameter of the null geodesic. The null geodesics are parametrised by only one parameter, namely the impact parameters b. To study the behaviour of the null geodesics, we may write equation (2.8) in an effective potential form with the effective potential given by

$$V = (\Lambda^4/r^2)(1-2m/r). \tag{2.9}$$

Propagation of zero mass particles is possible when $1/b^2$ exceeds V. For convenience we choose dimensionless units

$$x = r/m \qquad \beta = mB$$

and express V as a function of x,

$$V = x^{-2}(1-2/x)(1+\beta^2 x^2)^4.$$

The parameter β measures the strength of the magnetic field for a fixed mass m. The entire behaviour of the orbits of zero mass particles is dependent on the effective potential V. Therefore it is necessary to study the properties of the function V in detail.

3. The effective potential V and the orbits

Since we shall be interested in the region outside the event horizon ($r > 2m$) we shall mainly examine the behaviour of V in the region $x > 2$. The function $V(x)$ vanishes at $x = 2$ and tends to infinity $V(x) \sim \beta^8 x^6$ as x grows arbitrarily large. The existence of the maximum or the minimum of the potential plays a crucial role in determining the orbit, and hence we set $\mathrm{d}V/\mathrm{d}x = 0$. This yields the cubic

$$f(x) \equiv 3\beta^2 x^3 - 5\beta^2 x^2 - x + 3 = 0. \tag{3.1}$$

The roots of $f(x) = 0$ will be the points where the effective potential V possesses extrema. The cubic $f(x)$ possesses three roots of which one is real and negative. The other two roots of f could be complex or real depending on the value of β. This can be easily verified by checking whether the minimum value of f for $x > 2$ is negative. This is achieved again by setting $\mathrm{d}f/\mathrm{d}x = 0$. This equation is easily solved and it is seen that the minimum value of f is negative if the following condition holds:

$$(25\beta^2 + 9)[5\beta + (25\beta^2 + 9)^{1/2}] - 342\beta > 0. \tag{3.2}$$

For sufficiently small values of β the expression is clearly positive, its value being about 27. Therefore for these small values of β, V possesses two extrema, a maximum and a minimum, with the maxima occurring for a smaller value of x. For $\beta = 0.1$, say, the expression becomes negative and hence V does not possess any extrema in the relevant range of x. However, at large β the expression again changes its sign and V again possesses extrema, but they occur for $x < 2$. Therefore for sufficiently large values of β the function V is monotonically increasing in the range of interest $x > 2$. Therefore our results can be summarised as follows. There are essentially two types of behaviour of the effective potential V. For β less than a critical value, say β_c, the function V possesses a maximum followed by a minimum, while for $\beta > \beta_c$, V is monotonically increasing in the region of interest, $x > 2$. It remains to determine the value of β_c. Computations on the computer by the usual root finding methods show that

$$\beta_c = 0.094\,68. \tag{3.3}$$

The two types of behaviour of V are shown in figures 1(a) and 2(a). Figure 1(a) shows the graph of V as a function of x for $\beta = 0.07$ which is less than β_c. V possesses both a maximum and minimum in this case. Figure 2(a) depicts the other case where β is chosen to be 0.1 which is greater than β_c. The potential V in this case is monotonically increasing. As β increases further, the slope of V becomes steeper, which shows that the gravitational field increases rapidly with the increase in the magnetic field strength.

From the properties of the effective potential V discussed above, the behaviour of the orbits of the zero mass particles can be deduced. When $\beta < \beta_c$ the existence of the maxima and minima of V gives a structure of a potential well in the range of $x > x_{\max}$, where $x_{\max}$ is the value of x at which the maximum occurs. Since β_c itself is small compared with unity, the case is not too dissimilar from the Schwarzschild

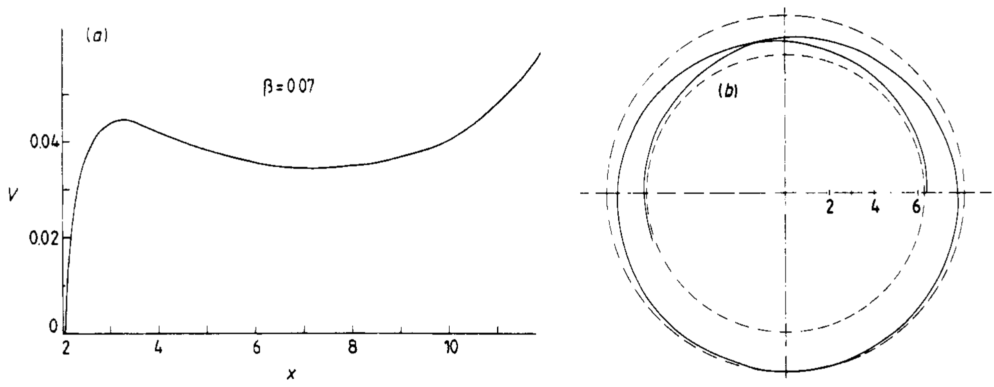

Figure 1. (a) The effective potential V is plotted against x for $\beta = 0.07$. Since $\beta < \beta_c$ the effective potential possesses both a maximum and a minimum. Persisting orbits, stable circular orbits exist in this case. (b) The full curve is the null trajectory of a particle for $b = 5.345$ m and $\beta = 0.07$. It is bound between the two broken circles of radii $x = 6.3$ and $x = 8.1$. This is a typical example of a persisting orbit.

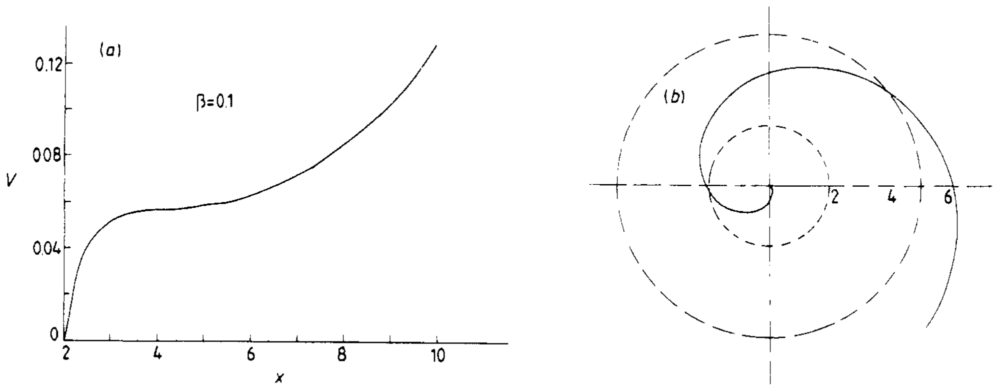

Figure 2. (a) The effective potential V is plotted against x for $\beta = 0.1$. As $\beta > \beta_c$ the effective potential is monotonically increasing for $x > 2$. There are no persisting orbits and every particle with non-zero impact parameter must fall into the blackhole singularity. (b) The full curve is the null trajectory of a particle which falls into the singularity $r = 0$, having $b =$ 3.5355 m and $\beta = 0.1$. The particle spirals into the singularity.

case where $x_{\max} = 3$. In this case $x_{\max} \sim 3$. If $1/b^2$ is less than $V(x_{\max})$ and greater than $V(x_{\min})$, where $x_{\min}$ is the value of x at which V possesses a minimum, then a persisting orbit exists and the particle is trapped in the potential well. It does not fall into the black hole or escape to infinity. In fact, if $b \neq 0$, that is if the particle propagates non-radially, it can never escape to infinity. This happens because $V(x)$ grows without bound for x tending to infinity and the particle must experience a turning point for a finite value of x. Figure 1(b) shows a persisting orbit per $\beta = 0.07$ and $b = 5.345$ m. The particle is bound between two circles with radii as the two roots of $1/b^2 - V(x) = 0$. The potential well is extremely shallow, with the result that the particle tends to circle the black hole considerably in moving from one bounding circle to the other bounding circle. For $1/b^2 \geqslant V(x_{\min})$

$$\varphi \sim [(1+\beta^2 x_{\min}^2)^4 / x_{\min}^2] \sin^{-1} [a(x - x_{\min})/\varepsilon]$$

where

$$\varepsilon^2 = 1/b^2 - V^2(x_{\min}) \qquad a^2 = \tfrac{1}{2} \mathrm{d}^2 V/\mathrm{d}x^2|_{x=x_{\min}}.$$

The particle may escape to infinity only if it travels radially outwards. The radial propagation is described by

$$\mathrm{d}r/\mathrm{d}\lambda \sim E/B^4 r^4.$$

One finds that $\mathrm{d}r/\mathrm{d}\lambda$ becomes increasingly small as r increases, implying that the particle finds it increasingly difficult in climbing the potential gradient. The gradient becomes steeper and the gravitational field strengthens as r increases since, broadly speaking, a sphere of larger radius r encloses more magnetic energy which produces the gravitational field.

If $1/b^2$ is greater than $V_{\max}$ the particle must end up in the black hole either by bouncing at the potential barrier or otherwise. At $x = x_{\min}$ there exists a stable circular orbit, while for $x = x_{\max}$ there is an unstable circular orbit. A slight perturbation in the orbit is sufficient to make the particle fall into the black hole.

In the case of $\beta > \beta_c$ there is no persisting orbit, and for $b \neq 0$ the particle must eventually end up at the $r = 0$ singularity. Figure 2(b) shows a typical trajectory of such a particle. There are no stable or unstable circular orbits as were present in the earlier case. The magnetic field is too strong to allow any such behaviour.

4. The weak magnetic field case $\beta \ll 1$

This is the case which may be of practical interest since $\beta \sim \mathrm{O}(1)$ involves very large masses and extremely powerful magnetic fields. The essential features of the orbits and the effective potential can be easily foretold since we are familiar with the Schwarzschild photon orbits. For not-too-large values of x the effective potential will somewhat mimic the Schwarzschild case. Only when x becomes sufficiently large will the $\beta^8 x^6$ term predominate and make the effective potential deviate from the $\beta = 0$ effective potential. Thus one may picture the effective potential as follows: Starting from the value zero at $x = 2$, $V(x)$ will attain a maximum of $\sim \frac{1}{27}$ at about $x_{\max} \sim 3$ and then fall rapidly until it is almost zero. At somewhat larger values of x the shape of the potential is like an extremely shallow well. Finally, when x becomes sufficiently large the potential will grow, and tend to infinity as x becomes arbitrarily large. With this broad picture in mind one can now work out the details of the essential features of the effective potential profile.

The maximum of V can be calculated on remembering that $x_{\max} = 3$ in the $\beta = 0$ case. Therefore, setting $x_{\max} = 3 + \varepsilon$, where ε is a small quantity, in $f(x) = 0$ and retaining only the first-order terms in ε and β^2, it is possible to solve for ε. The result is that $x_{\max} = 3 + 36\beta^2$. Therefore the maximum of the effective potential is 'pushed out' on application of the magnetic field. The unstable circular orbit possesses a larger radius than that of the Schwarzschild case. The height of the maximum is $\sim \frac{1}{27}(1 + 36\beta^2)$ which is higher than the case of $\beta = 0$. The impact parameter for the unstable circular orbit has the value $b \sim 3\sqrt{3}m(1 - 18\beta^2)$.

The minimum of V can be determined by observing that it occurs for a large value of x when β is small. The relation $f(x)=0$ yields in this case $x_{\min}\sim 1/\sqrt{3}\beta$. Therefore the stable circular orbit becomes larger and tends to infinity as β decreases to zero. The value of the minimum may be computed to yield $V_{\min}\sim\frac{256}{27}\beta^2$. In doing the calculations we have retained the first-order terms in β^2. The associated impact parameter of the particle to traverse this circular orbit is given by the value $3\sqrt{3}m\beta/16$. This situation is similar to that of the Melvin magnetic universe, where the effective potential is given by

$$V_{\mathrm{M}}(r)=r^{-2}(1+B^2r^2)^4. \tag{4.1}$$

Here we have fallen back on the original coordinates, as our dimensionless coordinates are undefined since $m=0$. The function $V_{\mathrm{M}}(r)$ possesses a minimum at $r=1/\sqrt{3}B$ with a height of $\frac{256}{27}B^2$. We observe the similarity between the effective potential V for large x and $V_{\mathrm{M}}(r)$ given by (4.1). This is due to the fact that the term $(1-2/x)$ occurring in $V(x)$ is reduced to unity for large x and V approaches V_{M} in the limit.

Therefore we note the following point. The weak field case of the Ernst space–time is very much like the Schwarzschild for small values of x since the term $(1+\beta^2x^2)^4$ is of the order of unity. However, when x becomes large the potential function makes a transition to that of the Melvin magnetic universe since the term $1-2/x$ approaches unity. This again highlights the fact that the Ernst space–time behaves like its limiting cases $m=0$ and $B=0$ in different regions of the radial coordinate.

5. Gravitational redshift

The redshift of a particle is of major concern in energy considerations. The energy flux profiles depend to a large extent on this redshift factor. We give here the computations of the redshift of a particle emitted near the black hole and observed far away from the object. We naturally expect that the strengthened gravitational field due to the magnetic field energy will increase the redshift factor. We assume that both the emission and the detection of the particle takes place in the static observer's frame of reference. The redshift is given by

$$1+z=\nu_{\mathrm{e}}/\nu_{\mathrm{o}}=(k^\alpha u_\alpha)_{\mathrm{e}}/(k^\alpha v_\alpha)_{\mathrm{o}} \tag{5.1}$$

where ν_{e} is the frequency of the emitted particle and ν_{o} the frequency at the observed point. The vector k^α is tangent to the null geodesic. The time-like vectors u^α and v^α are the unit tangents to the world lines of the static observers at the point of emission and observation point respectively. If the emission occurs at $r=r_1$ and the observation point is $r=r_2$ then the non-zero components of u^α and v^α are

$$u^0=[\Lambda_1(1-2m/r_1)^{1/2}]^{-1} \qquad v^0=[\Lambda_2(1-2m/r^2)^{1/2}]^{-1}$$

where

$$\Lambda_j=(1+B^2r_j^2)^4 \qquad j=1,2.$$

From the first integral (2.6) we have

$$1+z=\frac{\Lambda_2[1-(2m/r^2)]^{1/2}}{\Lambda_1[1-(2m/r_1)]^{1/2}}. \tag{5.2}$$

The above equation describes the redshift of the particle in the Ernst field. It is seen that the redshift depends crucially on the magnetic field (to its eight power). The dependence on the mass is not that striking. The expression reduces to the Schwarzschild case when $B = 0$.

6. Conclusion

We have examined the null geodesics in the Ernst space–time in the equatorial plane $\theta = \pi/2$. It is seen that the behaviour of the null geodesics is essentially of two types, depending on whether the magnetic field parameter β is greater or less than the critical value $\beta_c = 0.094\,68$. For $\beta < \beta_c$ we find that there exist persisting orbits, stable and unstable circular orbits. However, if $\beta > \beta_c$ none of these types of orbits are present and every particle with non-zero angular momentum must fall into the singularity $r = 0$. The particles with only pure radial motion can escape to infinity. For small values of β, that is, $\beta \ll 1$ we can explore further the behaviour of the effective potential. The radii of both the stable and unstable circular orbits are obtained in this case in the leading terms in β. Finally, the gravitational redshift has been computed which is of importance in the energetics of the particle.

Acknowledgment

The authors wish to express their gratitude towards Professor M A Melvin for his valuable observations. One of us (DNS) takes this opportunity to thank Professor J V Narlikar for the encouragment given to him for providing facilities and programme of work at the Tata Institute of Fundamental Research.

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